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SEMIGROUPS WITH A DENSE SUBGROUP

BY

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ABSTRACT. The purpose of this paper is twofold. First, it is shown that the ideal structure of a semigroup with dense subgroup is closely related to its transformation group structure. That is, if a left orbit through a given point is locally compact, then the members of this orbit are precisely those elements which generate the same left ideal as the given point.

Secondly, the author gives a number of theorems which have as their goal the establishment of a natural product structure near a nonzero idempotent. Specifically the work of F. Knowles [11] is improved upon to include (1) the possibility of a nonconnected group; (2) the possibility of a nonsimply connected orbit; and (3) the case in which the boundary of the group is more than a single orbit.

Introduction. Numerous papers ostensibly dealing with semigroups with identity on a manifold (and having their origins in [8]) have in reality dealt almost exclusively with the maximal group G and its closure. In fact, for many of the results so obtained, one may as well have assumed merely that one had a dense Lie subgroup. No exhaustive study has been made of this hypothesis, nor does this paper in any way attempt to summarize what has been done in this area. Results have been obtained, for example, concerning the question of when an arbitrary dense subgroup is open, and these results are totally ignored in the present paper.

The emphasis in the present paper is twofold. First, it is shown that the transformation group structure inherent in the semigroup is intimately connected with its structure via Green's relations.

Second, several results are obtained which make it possible to give a topological decomposition of the semigroup analogous to that obtained in [10]. Specifically, the following is a special case of Theorem 14 of the paper: Let S be a semigroup on a simply connected manifold. Suppose S has a completely simple kernel M and a dense connected Lie subgroup G. Let e be an idempotent in M

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with Ge, eG, and $eG_L(e)$ locally compact. Then S is homeomorphic to

$$[Se \cap E(S)] \times eSe \times P^- \times [eS \cap E(S)]$$

where $P = G_L(e) \cap G_R(e)$. It should be remarked that Theorem 14 also gives a significant generalization of Theorem 5 of [7]. For in the latter, Knowles considers only semigroups in which the boundary of G is a single orbit Ge. It is easily seen that the boundary in this case is a left group (see [1]) and hence is the completely simple kernel of S. Had Theorem 14 been available prior to work on [5], it could have made the work therein much easier.

Even if S has a zero, in which case Theorem 14 is of no value, one has at least a local version of the above product structure near nonzero idempotents. In fact, Theorem 11 handles as well the case in which S need not be simply connected nor have a completely simple kernel.

Notation. G will always be a dense topological subgroup of the semigroup S, and G_0 denotes the identity component of G. $L = \partial G$ denotes $S - G$, and e is an idempotent in L. $H = G_L(e)$ denotes the left isotropy group of e, and $K = G_R(e)$ denotes the right isotropy group of e. P will denote $H \cap K$.

We shall use $\mathcal{J}$, $\mathcal{L}$, and $\mathcal{R}$ to denote Green's relations: namely, $a \ \mathcal{J} \ b$ if and and only if a and b generate the same two-sided ideal. $\mathcal{L}$ and $\mathcal{R}$ are the one-sided analogues. J_a will denote the $\mathcal{J}$-class containing a. The meanings of L_a and R_a are analogous. $H(e)$ denotes the maximal subgroup containing e. $E(S)$ denotes the set of idempotents of S.

We let $A = \{g \mid ge \in H(e)\}$, $B = \{g \mid eg \in H(e)\}$, and $C = A \cap B$. Observe that $H \cap B = P$ and that $K \cap A = P$.

We begin with a theorem which links the ideal structure of S with the transformation group structure. We make use of the following well-known result from topology (see [2, p. 255, Exercise 7]): If X is a locally compact, dense subset of a space Y, then X is open in Y.

REMARK. Gx is dense in Sx. Thus if Gx is locally compact, it is open in Sx.

THEOREM 1. *Let G be a dense subgroup of S, and suppose $x \in S$ is such that Gx is locally compact. Then the orbit Gx coincides with the $\mathcal{L}$-class containing x. In particular, if $x \neq 0$ and if Sx is a 0-minimal left ideal, then $Gx = Sx - \{0\}$.*

PROOF. If $y \in Gx$, then $y \in Sx$ so $Sy \subseteq Sx$. But also $x \in Gy$ so $Sx \subseteq Sy$. Thus if $y \in Gx$, then $Sy = Sx$, so that $y \in L_x$. Conversely, suppose $y \in L_x$, so that $Sy = Sx$. Now Gy is dense in $Sy = Sx$. If $y \notin Gx$, then $Gy \cap Gx = \emptyset$, i.e.,

$Gy \subset Sx - Gx$. Since Gy is dense in Sx, this would give $Sx - Gx$ dense in Sx. But since Gx is locally compact and dense in Sx, Gx is open in Sx, so $Sx - Gx$ is closed in Sx. Thus $Sx - Gx = Sx$. But since $Gx \neq \emptyset$, this is impossible. Hence if $y \in L_x$, then $y \in Gx$, so that $L_x = Gx$.

Now suppose Sx is a 0-minimal left ideal of S. If $y \in Sx - \{0\}$, either $Sy = \{0\}$ or $Sy = Sx$. But since S has an identity, $Sy \neq 0$, so $y \in L_x = Gx$. Thus in this case $Gx = Sx - \{0\}$.

A frequent hypothesis below is that $H(e) = J_e \cap eSe$. We remark that this is true if Se is left minimal or left 0-minimal. Note also that if S is completely semisimple (see [1, vol. II, p. 32] for definition and notation), this property holds for *every* idempotent e. For certainly $H(e) \subset J_e \cap eSe$. Conversely, let ν: $J(e) \longrightarrow J(e)/I(e)$ be the natural map, and let $G^* = \nu(eSe)$. Then, since $J(e)/I(e)$ is completely 0-simple, G^* is a group with zero. Since $J_e = J(e) - I(e)$, $\nu(J_e) \cap G^*$ is a group, whence $J_e \cap eSe$ is a group. Since $H(e)$ is the maximal group, $H(e) = J_e \cap eSe$.

COROLLARY 2. *Let G be a dense subgroup of S, and let $e \in E(S)$ be such that $H(e) = J_e \cap eSe$ and such that Ge is locally compact. Then*

(1) *$H(e)$ is open in eSe,*

(2) *$H(e)$ is closed in $L_e = Ge$, and*

(3) *$H(e)$ is a topological group.*

PROOF. Since $H(e) \subset L_e \cap eSe \subset J_e \cap eSe$ (and similarly for R_e), if $H(e) = J_e \cap eSe$, then $H(e) = L_e \cap eSe = R_e \cap eSe$. (1) follows from the fact that $L_e = Ge$ is open in Se, and the fact that $eSe = Se \cap eS$. (2) follows from the fact that eSe is closed in S. To get (3), recall that $H(e)$ is closed in Ge and hence locally compact. By [3], any such group is a topological group.

PROPOSITION 3. *Let G be a dense subgroup of S, let e be any idempotent such that Ge and eG are locally compact, and let $A = \{g \in G: ge \in H(e)\}$. Then*

(1) *A is a subgroup of G,*

(2) *the map $g \longrightarrow ge$, with $g \in A$, is a homomorphism of A onto $H(e)$ with kernel $G_L(e)$, and*

(3) *if $H(e) = J_e \cap eSe$, then A is closed in G.*

PROOF. In [5], Horne defines $G^L(e) = \{g \in G: ge \in eG\}$, and observes that $G^L(e)$ is a subgroup, and that $G^L(e) \cdot e = Ge \cap eG$. He also observes that the latter is a group. We wish to see that $H(e) = Ge \cap eG$, so that $G^L(e) = A$. Since $Ge \cap eG$ is a subgroup of eSe, certainly $Ge \cap eG \subset H(e)$ since $H(e)$ is the (unique) maximal subgroup of eSe. On the other hand, it is well known (see, for example, [1]) that $H(e) \subset L_e \cap R_e$, which equals $Ge \cap eG$ by Theorem 1. Thus $H(e) = Ge \cap eG$. This establishes (1). The proof of (2) is straightforward and is omitted. For (3), recall that if $H(e) = J_e \cap eSe$, then $H(e)$ is closed in Ge (Corollary 2). If $f(s) = se$, then $A = f^{-1}(H(e))$, which is therefore closed.

We now seek to generalize Theorems 1 and 2 of [6], and also to give some natural conditions which put us in a situation to apply this generalized version of the latter.

THEOREM 4. *Let G be a locally compact, σ-compact group of finite dimension which is dense in a semigroup S. If e is an idempotent such that Ge is locally compact, then $e \in G_L(e)^-$.*

PROOF. We need only inspect Knowles' proof to discover that the only time connectivity of G is used is to get Ge homeomorphic to $G/G_L(e)$. From Theorem I.2.5 of [4], this will be true provided G is locally compact, σ-compact and provided Ge is locally compact.

THEOREM 5. *Let G be a dense subgroup of a semigroup S, and let e be an idempotent. Suppose V is a subspace of G such that the map $v \longrightarrow ve$ is a homeomorphism of V onto Ge. Then the map $m\colon V \times G_L(e) \twoheadrightarrow G$ is a homeomorphism. Moreover, if $\partial G = Ge$, then*

(i) *$G_L(e)^- = G_L(e) \cup \{e\}$, and*

(ii) *$m\colon V \times G_L(e)^- \twoheadrightarrow G^-$ is a homeomorphism.*

PROOF. Since $Ve = Ge$, certainly $G = V \cdot G_L(e)$. Suppose $v_1h_1 = v_2h_2$ with $v_i \in V$, $h_i \in G_L(e)$. Then $v_1e = v_1h_1e = v_2h_2e = v_2e$, so $v_1 = v_2$. Then $h_1 = h_2$, so $m\colon V \times G_L(e) \twoheadrightarrow G$ is 1-1 and onto. If $v_ih_i \longrightarrow vh$, then $v_ie \longrightarrow ve$, so $v_i \longrightarrow v$, whence $h_i \longrightarrow h$. This establishes the first statement. The remainder of the theorem is a generalization of Theorem 2 of [6]. The only nonobvious ingredient of Knowles' proof is the fact that $e \in G_L(e)^-$. To see this, let $v_ih_i \longrightarrow e$; then $v_ie \longrightarrow e$, so $v_i \longrightarrow 1$, and $h_i \longrightarrow e$.

LEMMA 6. *Let G be a dense subgroup of S, and let M be a completely 0-simple ideal in ∂G. Suppose there is an element $x \neq 0$ in M such that Gx and xG are locally compact. Then if $y \in M - \{0\}$, Gy and yG are homeomorphic to Gx and xG, respectively, and in particular are locally compact.*

PROOF. By Theorem 6.25 of [1], $Sx = Mx$ is a 0-minimal left ideal of S. Thus by Theorem 1, $Sx = Gx \cup \{0\}$. Since $Sx \cap yS \neq 0$ [1], $Gx \cap yS \neq \varnothing$. Let $z \in Gx \cap yS$. Then $z = gx$ for some $g \in G$, so $zG = (gx)G = g(xG)$ which is homeomorphic to xG. Thus zG is locally compact, so $zG = zS - \{0\}$ by the dual to Theorem 1. Since $z \in yS$, $zS = yS$, so $y \in zS - \{0\} = zG$, so $yG = zG$ is locally compact, and homeomorphic to xG. Dually, Gy is homeomorphic to Gx.

The following is of interest in its own right, as well as being an ingredient of what is to come.

PROPOSITION 7. *Let G be a locally compact, σ-compact group of finite dimension, which is dense in a semigroup S. Let M be a completely 0-simple ideal in ∂G, and let e be a nonzero idempotent in M such that Ge and eG are locally compact. Then $M \cap \partial H = eG \cap E(S)$, where $H = G_L(e)$. In particular, if $M = \partial G$, then $\partial H = eG \cap E(S)$.*

PROOF. Let $f \in eG \cap E(S)$. Since $f \in eG$, $G_L(f) = G_L(e)$. Moreover, Gf is locally compact by the above lemma, so $f \in G_L(f)^- = G_L(e)^-$ by Theorem 4. Certainly $f \in M$; hence $eG \cap E(S) \subseteq M \cap \partial(G_L(e))$.

Conversely, let $x \in M \cap \partial H$. Since e is a right zero for H^-, $xe = e$, so $e \in xS$, which implies that $eS \subseteq xS$. Since $x \in M$, xS is a 0-minimal right ideal, so $eS = xS$, so $x \in eS$. Since e is a left identity for eS, $ex = x$. But then $x^2 = (ex)(ex) = e(xe)x = e^2x = ex = x$, so x is idempotent. Hence $x \in eS \cap E(S)$. Since $eG = eS - \{0\}$ (Theorem 1) and since $x \neq 0$, $x \in eG \cap E(S)$. Thus $M \cap \partial H = eG \cap E(S)$.

Terminology. We say W is a local cross section (l.c.s.) to H in G, provided the map $m: W \times H \longrightarrow G$ is a homeomorphism onto a neighborhood of 1 in G.

LEMMA 8. *Let S be a semigroup satisfying the hypotheses of Theorem 4. Then $H^- = \{s | se = e\}$. Moreover, if eH is locally compact, then $P^- = \{s | se = es = e\}$, where $P = G_L(e) \cap G_R(e)$.*

PROOF. Let W be a local cross section to H in G. Then the map $w \longrightarrow we$ is a homeomorphism of W onto a neighborhood of e in Ge. Clearly $H^- \subset \{s | se = e\}$. Now suppose $se = e$. Let $g_i \longrightarrow s$, $g_i \in G$. Then $g_ie \longrightarrow se = e$, so eventually $g_ie = w_ie$ for some $w_i \in W$. Let $h_i = w_i^{-1}g_i$. Then $h_i \in H$ and $g_i = w_ih_i$. Since $w_ie = g_ie \longrightarrow e$, $w_i \longrightarrow 1$ so $h_i \longrightarrow s$. Thus $s \in H^-$, as desired.

The second statement is an application of the first to the semigroup $G_L(e)^-$, with H acting on ∂H on the right. The isotropy for this action is $G_L(e) \cap G_R(e) = P$, so by what we have just shown, $P^- = \{s \in G_L(e)^-: es = e\}$. Clearly the latter set is $\{s: se = es = e\}$.

LEMMA 9. *Suppose S is a semigroup, and $e \in E(S)$ is such that $H(e)$ is an open topological subgroup of eSe. Then the multiplication map m: $[L_e \cap E(S)] \times H(e) \longrightarrow Se$ is an isomorphism onto a neighborhood of e in Se.*

PROOF. Observe that $L_e \cap E(S)$ is a left trivial semigroup. Let a_i, $a \in L_e \cap E(S)$, and let x_i, $x \in H(e)$. Suppose $a_i x_i \longrightarrow ax$. Multiplication by e on the left gives $x_i \longrightarrow x$. Since $H(e)$ is a topological group, $x_i^{-1} \longrightarrow x^{-1}$, so $a_i = a_i e \longrightarrow ae = a$. Similarly, if $a_1 x_1 = a_2 x_2$, $x_1 = x_2$ so $a_1 = a_2$. Thus the map m: $[L_e \cap E(S)] \times H(e) \longrightarrow Se$ is a homeomorphism.

The proof that the map is algebraically an isomorphism is trivial and is omitted.

To see that the image is a neighborhood of e in Se, let $s_i e \longrightarrow e$. Then $es_i e \longrightarrow e$, so eventually $es_i e \in H(e)$, since $H(e)$ is open in eSe. Moreover, we wish to see that $s_i e(es_i e)^{-1} \in L_e \cap E(S)$. Certainly $s_i e(es_i e)^{-1} \in E(S)$. To see that $s_i e(es_i e)^{-1}$ generates Se, it suffices to show that $e \in S[s_i e(es_i e)^{-1}]$. Since $e = e[s_i e(es_i e)^{-1}]$, this is clear. Thus $s_i e(es_i e)^{-1} \in L_e \cap E(S)$. Since $s_i e = s_i e(es_i e)^{-1} es_i e$, we are done.

LEMMA 10. *Suppose S is a semigroup having a dense subgroup which is locally compact, σ-compact, and finite dimensional. Suppose $e \in \partial G$ is such that $H(e) = J_e \cap eSe$, and such that Ge and eG are locally compact. Then the following are equivalent*:

(1) *eH is locally compact,*

(2) *Ke is locally compact,*

(3) *BH is open in G,*

(4) *KA is open in G.*

Moreover, if these conditions are satisfied, then

(5) *eH is open in $eS \cap E(S)$,*

(6) *Ke is open in $Se \cap E(S)$,*

(7) *Ce is open in $Ae = H(e)$.*

PROOF. (1) $\Rightarrow$ (3) *and* (5). First observe that $eG \cap E(S) = R_e \cap E(S)$ is right trivial and open in $eS \cap E(S)$. Since every member of $eG \cap E(S)$ is in H^- (Lemma 8), it is easily seen that eH is dense in $eG \cap E(S)$. It follows that eH is open in $eG \cap E(S)$, and hence in $eS \cap E(S)$. Now $eB = H(e)$, so by the previous lemma it follows that $eBeH = eBH$ is open in eS and hence in eG. Thus BH is open in G.

(3) $\Rightarrow$ (4) *and* (7). Recall that we set $C = A \cap B$. Since BH is open in G, $A \cap BH = CH$ is open in A. Since the map $g \longrightarrow ge$ is open [3], this means $CHe = Ce$ is open in $Ae = H(e)$. Now observe that KA is open in G if and only if KAe is open in Ge. Thus we need to show that if $g_i e \longrightarrow e$, eventually $g_i e \in KAe$. Since $BHe = Be$ is open in Ge, we may as well assume $g_i \in B$. By the lemma, eventually $g_i e \in [Ge \cap E(S)] \cdot Ce$. We abbreviate $g_i e$ by ge and write $ge =$

sce where $s \in Ge \cap E(S)$ and $c \in C$. Let $k = gc^{-1}$. We wish to see that $k \in K$. But $ek = egc^{-1} = egec^{-1}e = escec^{-1}e = ese = e$ since both g and c are in B. Thus $k \in K$, as desired. Then $ge = kce \in KCe \subset KAe$, so KAe is open in Ge, whence KA is open in G.

(4) ⇒ (2) *and* (6). As we have seen, KA is open in G if and only if $KAe = KeAe$ is open in Ge. Since $Ke \subset Ge \cap E(S)$ and $Ae = H(e)$, it follows from the previous lemma that KAe is homeomorphic to $(Ke) \times (Ae)$. Thus if KAe is locally compact, so is Ke.

(4) ⇒ (6) is identical to (1) ⇒ (5).

The remaining equivalences follow by dualizing the above.

The following theorem gives us the local product structure referred to in the introduction.

THEOREM 11. *Let S be a semigroup satisfying the hypotheses of the lemma, and any of conditions* (1)–(4) *therein. Then there is a neighborhood of e in S locally homeomorphic to* $[Ge \cap E(S)] \times H(e) \times P^- \times [eG \cap E(S)]$, *or alternately, to* $K/P \times A/H \times P^- \times H/P$.

PROOF. We showed in the lemma that Ke is open in $Ge \cap E(S)$ and that eH is open in $eG \cap E(S)$. It follows that K/P and H/P are locally homeomorphic to $Ge \cap E(S)$ and $eG \cap E(S)$, respectively. Certainly A/H is isomorphic to $H(e)$ by Proposition 3.

Now let R be a local cross section (l.c.s.) of P in K, let T be a l.c.s. of $P = H \cap B$ in $C = A \cap B$, and let Q be a l.c.s. of P in H. Then the maps $r \longrightarrow re$ $(r \in R)$ and $q \longrightarrow eq$ $(q \in Q)$ are homeomorphisms onto open subsets of $Ge \cap E(S)$ and $eG \cap E(S)$, respectively. Moreover, since Ce is open in $Ae = H(e)$ (Lemma 10), C/P is isomorphic to Ce (an open subgroup of $H(e)$), to the map $t \longrightarrow te$ $(t \in T)$ is likewise a homeomorphism of T onto an open subset of $H(e)$. We shall show that the multiplication map m: $R \times T \times P^- \times Q \longrightarrow S$ is a homeomorphism onto a neighborhood of e in S.

(1) To see that m is a homeomorphism, it suffices to see that if $r_it_ip_iq_i \longrightarrow rtpq$ (with $r_i, r \in R$; $t_i, t \in T$; $p_i, p \in P^-$; and $q_i, q \in Q$), then $r_i \longrightarrow r$, $t_i \longrightarrow t$, $p_i \longrightarrow p$, and $q_i \longrightarrow q$. (The case in which all these nets are constant yields the one-to-oneness of m.) Recall that $R \subset G_R(e)$, $Q \subset G_L(e)$, $P = G_L(e) \cap G_R(e)$, and $T \subset A \cap B$. Thus $r_it_ie = r_it_ip_iq_ie \longrightarrow rtpqe = rte$, so that $et_ie = er_it_ie \longrightarrow erte = ete = te$. But then, since t_ie, $te \in H(e)$, $(et_ie)^{-1} = (t_ie)^{-1} \longrightarrow (te)^{-1}$ so $r_ie \longrightarrow re$, whence $r_i \longrightarrow r$. Likewise $t_i \longrightarrow t$ so $p_iq_i \longrightarrow pq$. Therefore $eq_i \longrightarrow eq$ so $q_i \longrightarrow q$ so $p_i \longrightarrow p$, as desired.

(2) We next show that $R \cdot T \cdot P^- \cdot Q$ is a neighborhood of e in S by showing that if $s_i \longrightarrow e$, eventually $s_i \in R \cdot T \cdot P^- \cdot Q$. So suppose $s_i \longrightarrow e$. Then

eventually $es_ie \in Te$, say $es_ie = t_ie$. Moreover, $s_i(es_ie)^{-1} \in Se \cap E(S)$ and $(es_ie)^{-1}s_i \in eS \cap E(S)$, so likewise, eventually $s_i(es_ie)^{-1} \in Re$ and $(es_ie)^{-1}s_i \in eQ$. Let us set $s_i(es_ie)^{-1} = r_ie$ and $(es_ie)^{-1}s_i = eq_i$. Henceforth we assume i is large enough so that all the above containments hold. We shall drop the subscript i for convenience. Then $se = re(ese) = rete = rte$ so $t^{-1}r^{-1}s \in \{s \mid se = e\} = G_L(e)^-$ by Lemma 8. Let $p = t^{-1}r^{-1}s$. Then $ep = et^{-1}r^{-1}s = et^{-1}er^{-1}s = (ese)^{-1}es = (ese)^{-1}s = eq$, so likewise $pq^{-1} \in G_R(e)^-$. Let $p_1 = pq^{-1}$. Since both p and q belong to $G_L(e)^-$, so does p_1. Thus $p_1 \in P^-$ by Lemma 8. Then $p = p_1q = t^{-1}r^{-1}s$ so $s = rtp_1q \in R \cdot T \cdot P^- \cdot Q$, as desired.

LEMMA 12. *Let S be a semigroup satisfying the hypotheses of the previous theorem. Suppose further that S is locally arc-wise connected. Then there is an arc in P^- running from 1 to e.*

PROOF. By the theorem, P^- is locally arc-wise connected at e. Thus there is an arc I in P^- running from e to some point x of P. Since e is a zero for P^-, $x^{-1}I$ is an arc in P^- running from e to 1.

Notation. With S as above, the arc guaranteed by the lemma will be denoted $[1, e]$.

LEMMA 13. *Let G be a locally connected group and H a connected, normal subgroup such that G/H is connected. Then G is connected.*

PROOF. Let $\nu: G \longrightarrow G/H$ be the natural map. Then $\nu(G_0)$ is an open (hence closed) subgroup of G/H, so $\nu(G_0) = G/H$. Then $G = G_0 \cdot H$. But since H is connected, $H \subset G_0$, so $G = G_0$.

Notation. Let G_0 denote the identity component of G. H_1 will denote $H \cap G_0$, $K_1 = K \cap G_0$, and $P_1 = P \cap G_0$. Likewise let $A_1 = A \cap G_0$, $B_1 = B \cap G_0$, and $C_1 = C \cap G_0$.

THEOREM 14. *Let S be a locally arc-wise connected, simply connected semigroup with a completely simple kernel M. Suppose further that S has a dense Lie subgroup G which is σ-compact (i.e. has a countable number of components). Let e be an idempotent in $M \cap \partial G_0$ such that Ge, eG, and eH are locally compact. Then S is homeomorphic to $[Se \cap E(S)] \times eSe \times P^- \times [eS \cap E(S)]$.*

PROOF. Since M is completely simple, $M = SeS$, and Se and eS are minimal left and right ideals, respectively [1]. Thus by Theorem 1, $Ge = Se$ and $eG = eS$. Moreover, Se and eS are retracts of S and hence are connected and simply connected. We wish to see that $G_0e = Ge$ (and likewise $eG_0 = eG$). Observe that G_0e is open in Ge since G_0 is open in G. In fact, for each $x \in G$, xG_0e is open in Ge. Also, if $y \in G$ and $yG_0e \cap xG_0e \neq \emptyset$, then $G_0ye \cap G_0xe = \emptyset$ (since

$yG_0 = G_0y$ and $xG_0 = G_0x$), whence $yG_0e = G_0ye = G_0xe = xG_0e$. Thus the sets xG_0e $(x \in G)$ are equal or disjoint. Since Ge is connected, it follows that $G_0e = Ge$. Thus G_0 is a connected group acting with simply connected orbit. By [9], this means H_1 is connected. Likewise, K_1 is connected.

Since $J_e = M$ and $H(e) = eSe \subset M$, certainly $H(e) = J_e \cap eSe$. Thus by Corollary 2, eSe is a topological group. Now by [10], M is homeomorphic to $[Se \cap E(S)] \times eSe \times [eS \cap E(S)]$. Since M is connected, so are each of $Se \cap E(S)$, eSe, and $eS \cap E(S)$. Also, eSe is simply connected, as a retract of Se.

By Lemma 10, eH is open in $eS \cap E(S)$. Likewise, Ke is open in $Se \cap E(S)$. Moreover, every member of $eS \cap E(S)$ is in eG and so has H as its isotropy group. Thus every orbit fH $(f \in eS \cap E(S))$ is open in $eS \cap E(S)$. Since $eS \cap E(S)$ is connected, $eH = eS \cap E(S)$. Likewise, $Ke = Se \cap E(S)$. Also by Lemma 10, Ce is an open (hence closed) subgroup of $H(e) = eSe$, so we have $Ce = eSe$.

Now $Ce = eSe$ is a factor of $Ge = [Se \cap E(S)] \times eSe$, and hence is locally compact. Thus the map $c \longrightarrow ce$ is open. Since $C_1 = C \cap G_0$ is open in C, C_1e is an open (hence closed) subgroup of eSe, so as above $C_1e = eSe$.

We wish to see that Ke and eH are contractible. Define F: $[1, e] \times Ke \longrightarrow Ke$ by $F(p, ke) = pke$. (Observe that $pke \in Se \cap E(S) = Ke$.) Then F is a homotopy from the identity map on Ke to the constant $\{e\}$, so Ke is a contractible manifold. Dually, eH is contractible. Thus by [11] there is a global cross section R of P_1 in K_1, and we write $K_1 = RP_1$. Dually $H_1 = P_1Q$ for some cross section Q of P_1 in H_1.

To get a global cross section of $P_1 = H_1 \cap B_1$ in C_1, we must show that the latter is connected. To see this, recall that $eC_1 = H(e)$, so that $K_1C_1 = B_1$. Also observe that B_1 is a Lie subgroup since B_1 is open in B which is closed in G. But then $B_1e = K_1C_1e = K_1eC_1e = [Se \cap E(S)] \cdot eSe = Ge$ is simply connected. Since $eSe = B_1/K_1$ is connected and since K_1 is connected, this means B_1 is connected by Lemma 13. But then $B_1 \cap H_1 = P_1$ is connected (since B_1e is simply connected). Since $eSe = C_1/P_1$ is connected, applying Lemma 13 again we see that C_1 is connected. Thus by [7], there is a global cross section T of P_1 in C_1.

Then the maps $r \longrightarrow re$, $t \longrightarrow te$, and $q \longrightarrow eq$ are homeomorphisms of R onto $Se \cap E(S)$, of T onto eSe, and of Q onto $eS \cap E(S)$, respectively. By Theorem 5, it follows that R is a cross section of P in K, that T is a cross section of P in C, and that Q is a cross section of P in H. Thus the local homeomorphism constructed in the proof of Theorem 11 is actually a global homeomorphism in the present case. We wish to see that the image is all of S (not merely a neighborhood of e in S). Let $s \in S$. Now $Re = Se \cap E(S)$, $Te = eSe$, and $eQ = eS \cap E(S)$. Let $ese = te$, $s(ese)^{-1} = re$, and $(ese)^{-1}s = eq$, where $t \in T$, $r \in R$, and $q \in Q$. Exactly as in Theorem 11 (proof, part (2)), $s = rtp_1q \in R \cdot T \cdot P^- \cdot Q$, where $p_1 = t^{-1}r^{-1}sq^{-1}$. Thus $S = R \cdot T \cdot P^- \cdot Q$, which is homeomorphic to $[Se \cap E(S)] \times eSe \times P^- \times [eS \cap E(S)]$. This concludes the proof.

It should be remarked at this point that if S has a 0, then necessarily $M = \{e\} = \{0\}$ in the statement of the theorem. In this case, of course, $P = G$ and the theorem is trivial. Nevertheless, this theorem appears to have wide application of a nontrivial nature. In particular, one possible application is for semigroups on a half-n-space of arbitrary dimension (see [5] for definition). The author will show the relevance of this theorem in a sequel to the present paper.

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